GEOMETRICAL ACOUSTICS IN A HETEROGENEOUS ANISOTROPIC ELASTIC SOLID: APPLICATION TO A WAVY COMPOSITE

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INTRODUCTION

Karal and Keller [1] developed the geometrical acoustics for wave propagation in a heterogeneous isotropic medium, generally adopting the methods used in geometrical optics [2,3]. It is very difficult to find a solution for wave propagation in a heterogeneous anisotropic medium. Here, instead of finding an exact solution, we extend the geometrical acoustics to a heterogeneous anisotropic medium to untangle the behavior of wave fronts spreading into an undisturbed region. The eikonal equation which contains information of the phase and group velocities, along with the transport equation which governs the amplitude of propagating waves, are derived. For a one-dimensionally heterogeneous anisotropic solid, wave propagation is two dimensional and it is possible to obtain closed-form analytic formulas for the ray path and travel time of a ray. These formulas are applied to find the path and travel time of rays generated from a pointlike source and detected by a small detector. The predicted arrival times agree well with observed values.

THEORY

Consider the following wave equation of hyperbolic type with no damping

$$\frac{\partial}{\partial x_j}\left[C_{ijkl}(\mathbf{r})\frac{\partial u_k}{\partial x_l}\right]-\rho(\mathbf{r})\frac{\partial^2 u_i}{\partial t^2}=0, \qquad (i,j,k,l=1,2,3) \tag{1}$$

where C_{ijkl} are the elastic constants of a solid medium, ρ is the density, $\mathbf{r}$ represents a space coordinate, t is a time variable, x_j denotes jth component of $\mathbf{r}$, and $\mathbf{u}$ is a displacement of wave motion. The solution of Eq. (1) in general comprises evolution of cone $t = \tau(\mathbf{r})$ in the four-dimensional space-time. We consider a wave front surface $\varphi(\mathbf{r},t) = t - \tau(\mathbf{r}) = 0$ moving along the characteristic curves called the rays in the space-time. The displacement $\mathbf{u}$ undergoes a sudden jump at the wave front $\varphi = 0$. The discontinuity singularities in the Fourier analysis consist of high frequency components and the behaviors in geometrical acoustics hold valid in the high-frequency approximation.

The projection of $\varphi(\mathbf{r},t)$ at a particular time t_o onto the space domain is a conical surface called the wave front $\tau(\mathbf{r}) = t_o$, which can be physically interpreted as the arrival time of the wave emanating from a source at origin to reach a point $\mathbf{r}$. The projection of $\tau(\mathbf{r}) = t_o$ onto the two dimensional xy plane is a wave front curve whose schematics is drawn in Figure 1 at times t and $t + dt$., where the directions of wave normal and ray are specified by unit vectors $\mathbf{n}$ and $\mathbf{l}$, respectively. The projection of the characteristic curves or rays onto the xy plane is also shown in the figure. Let's denote the phase velocity of a wave surface $\tau(\mathbf{r})$ moving along the direction of wave normal $\mathbf{n}$ by υ and denote the inverse phase velocity or slowness by $\mathbf{p}$, which is expressed as $\mathbf{p} \equiv \mathbf{n}/\upsilon = \nabla\tau(\mathbf{r})$. The group velocity $\mathbf{V}_g \equiv d\mathbf{r}/dt$ or the velocity of a ray, phase velocity, and slowness are related by the relations $\mathbf{V}_g \cdot \mathbf{n} = \upsilon$ and $\mathbf{V}_g \cdot \nabla\tau = \mathbf{V}_g \cdot \mathbf{p} = 1$.

In the region inside $\tau(x,y) = t$, $\varphi > 0$ and outside $\tau(x,y) = t$, $\varphi < 0$. $\tau(x,y) = t$ or the wave

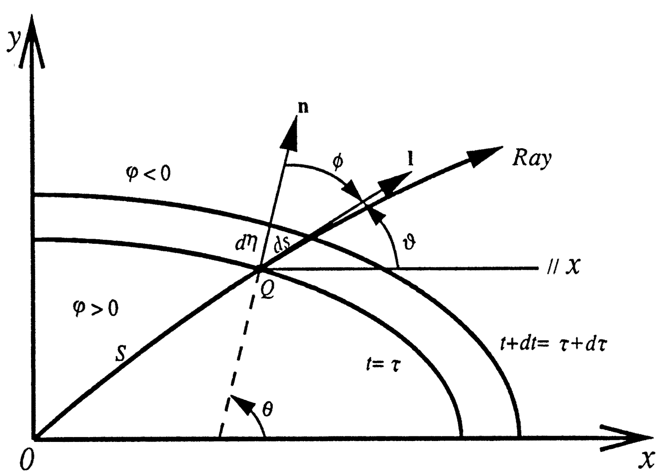

Figure 1. Two wave surfaces at times t and $t + dt$ with directions of wave normal and ray.

front $\varphi(\mathbf{r},t) = 0$ marks a boundary between a disturbed region ($\varphi > 0$) and an undisturbed region ($\varphi < 0$). Discontinuities in $\mathbf{u}$ suddenly take place at the boundary $\varphi(\mathbf{r},t) = 0$. We seek an ansatz or a trial solution

$$\mathbf{u} = \mathbf{A}_n(\mathbf{r}) S_n(\varphi), \quad (n=0,1,2,\cdots\infty) \tag{2}$$

where $S_n(\varphi)$ satisfies the following relation

$$S_n'(\varphi) = \frac{dS_n}{d\varphi} = S_{n-1}(\varphi). \quad (n \geq 1) \tag{3}$$

EIKONAL AND TRANSPORT EQUATIONS

We write u_i ($i = 1,2,3$) component in Eq. (2) as $u_i = A_{ni}(\mathbf{r})\ S_n[\varphi(\mathbf{r},t)]$, denote a derivative of Q with respect to a spatial variable x_j by $Q_{,j}$ ($j = 1,2,3$) and express a time derivative of Q as Q_t, where Q is an arbitrary differentiable variable. Substituting Eq. (2) into Eq. (1) and using Eq. (3), one obtains

$$\begin{aligned} &S_{n-2}\left[\left(C_{ijkl}\varphi_{,j}\varphi_{,l} - \rho\,\varphi_t^2\delta_{ik}\right)A_{nk}\right] \\ &+ S_{n-1}\left[C_{ijkl}\left(\varphi_{,j}A_{nk,l} + \varphi_{,l}A_{nk,j} + \varphi_{,lj}A_{nk}\right) + C_{ijkl,j}\varphi_{,l}A_{nk} - \rho\,\varphi_{tt}A_{ni}\right] + S_n\left(C_{ijkl}A_{nk,lj} + C_{ijkl,j}A_{nk,l}\right) = 0 \end{aligned} \quad .(4)$$

Note that $\varphi_{,j} = -\,\tau_{,j}$; $\varphi_t = 1$; $\varphi_{tt} = 0$. For $n = 0$ in Eq. (4), the coefficient of S_{-2} is required to be zero, resulting in

$$\Lambda = \det\left|C_{ijkl}\frac{\partial\tau}{\partial x_j}\frac{\partial\tau}{\partial x_l} - \rho\,\delta_{ik}\right| = \det\left|C_{ijkl}\,p_j\,p_l - \rho\,\delta_{ik}\right| = 0, \tag{5}$$

$$\Phi = \det\left|C_{ijkl}\,n_j\,n_l - \rho\,\upsilon^2\,\delta_{ik}\right| = 0 \ . \tag{6}$$

Eqs. (5) and (6) are the equations of slowness surface and phase velocity, respectively. The first part of Eq. (5) involving the wave front τ is known as the eikonal equation in geometrical acoustics. It can be readily shown from $\mathbf{V}_g \cdot \nabla\tau = \mathbf{V}_g \cdot \mathbf{p} = 1.$ and Eq. (5) that $\mathbf{V}_g = \nabla_{\mathbf{p}}\Lambda / \left(\mathbf{p}\cdot\nabla_{\mathbf{p}}\Lambda\right)$,which indicates that the direction of local group velocity or ray is normal to the slowness surface at that local point.

For $n = 0$ in Eq. (4), the coefficient of $S_{-1} = 0$ yields

$$\left(C_{ijkl}\,p_l\right)_{,j} A_{0k} + C_{ijkl}\left(p_j\,A_{0k,l} + p_l\,A_{0k,j}\right) = 0. \tag{7}$$

The above equation is known as the transport equation governing the amplitude of an elastic wave. From the requirement that the coefficient of $S_{n-1} = 0$, we proceed in a similar way to obtain the transport equation for $n \geq 1$

$$\left(C_{ijkl}\, p_l\right)_{,j} A_{nk} + C_{ijkl}\left(p_j\, A_{nk,l} + p_l\, A_{nk,j}\right) = C_{ijkl}\, A_{(n-1)k,lj} + C_{ijkl,j}\, A_{(n-1)k,l}. \quad (n \geq 1) \tag{8}$$

For detail of the derivation of Eqs. (4)-(8) refer to Ref. 4.

WAVE PROPAGATION IN TWO DIMENSIONS

Eq. (5) is a form of the nonlinear first-order partial differential equation and it is very difficult to obtain its solution in the general direction of propagation. However, when the elastic constants C_{ijkl} and the density ρ are the functions of one space variable, say x, it can be shown that the wave normals **n** or the slowness **p** are confined to a plane parallel to the x axis [4] and that the Snell's refraction law holds along the ray path. Denoting the angles between **n** and the x direction at various points by θ_i ($i = 0,1,2,3\cdots$), the Snell's law can be written as

$$\frac{\sin\theta_o}{\upsilon_o} = \frac{\sin\theta_1}{\upsilon_1} = \frac{\sin\theta_2}{\upsilon_2} = \cdots = h(\text{constant}). \tag{9}$$

We turn our attention to the wave propagation in the xy plane where both wave normals **n** and ray directions **l** are confined, as shown in Fig. 1. The directions of **n** and **l** are specified by angles θ and ϑ, respectively measured to the x axis. We define an angle $\phi \equiv \vartheta - \theta$, which is negative in Fig. 1. We also denote a differential arc length along the ray path between two wave fronts $\tau = t$ and $\tau + d\tau = t + dt$ as ds and a distance along the direction of a wave normal between two wave fronts by $d\eta$. $d\eta = \upsilon\, dt = ds \cos\phi$. The slowness surface Λ in a two dimensional case can be expressed as

$$\Lambda = p\upsilon - 1 = 0. \tag{10}$$

Referring to Ref. 4 and denoting υ' by $\partial\upsilon/\partial\theta$, the travel time τ and ray path of a wave which emanates from a source at the initial point (x_o, y_o) and arrives at a point (x_1, y_1), are given by

$$\tau = \int_{x_o}^{x_1} \frac{\cos\phi}{\upsilon\cos(\phi+\theta)}\, dx = \int_{x_o}^{x_1} \frac{dx}{\upsilon\cos\theta - \upsilon'\sin\theta}, \tag{11}$$

$$y_1 = y_o + \int_{x_o}^{x_1} \frac{\upsilon\sin\theta + \upsilon'\cos\theta}{\upsilon\cos\theta - \upsilon'\sin\theta}\, dx. \tag{12}$$

A wavy composite material is fabricated where reinforcing graphite fibers are imbedded in a wavy pattern in epoxy matrix. The fiber waviness is confined in the xy plane with the mean fiber direction along the x axis. The waviness is characterized by a sine wave form with amplitude 2 mm and periodic length 40 mm. A local fiber direction is specified by the angle β_f the x axis makes to the fiber and it satisfies the relation

$$\tan\beta_f = (\pi/10)\cos(\pi x/20). \tag{13}$$

The wavy composite is uniform along both z and y directions. As a result, both anisotropy and heterogeneity of the wavy composite is a function of one variable x only. We expect from the results of the previous section that the wave normals which are initially directed in the xy plane are confined on the xy plane and the Snell's law Eq. (9) holds valid for the wavy composite. A composite material which has the fibers running straight in the x direction ideally has transversely isotropic symmetry about the x axis. However, because of a less-than-ideal fabrication procedure, it is better characterized as possessing weak orthorhombic symmetry with very close proximity to transversely isotropy. Let's denote the x, y, and z axes by the directions 1, 2, and 3, respectively. Then, the composite specimen with straight fibers aligned in the x direction is characterized by nine elastic constants, which are measured to be C_{11} = 130.3 GPa, C_{22} = 11.00 GPa, C_{33} = 12.46 GPa, C_{44} = 2.95 GPa, C_{55} = 5.47 GPa, C_{66} = 4.96 GPa, C_{12} = 6.01 GPa, C_{13} = 1.04 GPa, and C_{23} = 6.27 GPa. The density of the wavy composite is 1524 kg/m^3.

Consider a small element of the wavy composite at a typical local point (x, y_o, z_o). The small local element is considered to have orthorhombic symmetry whose x and y symmetry axes are rotated about the third symmetry axis $z_o z$ by the angle β_f from the x axis. Then, the angle α which the wave normal $\mathbf{n}$ makes from the local fiber direction is

$$\alpha(x,\theta) = \theta - \beta_f = \theta - \tan^{-1}\left(\frac{\pi}{10}\cos\frac{\pi x}{20}\right), \tag{14}$$

where θ is the angle between the directions of $\mathbf{n}$ and x axis. The eikonal and phase velocity equations, Eqs. (5) and (6), hold at a local point (x, y_o, z_o) for waves propagating with wave normal $\mathbf{n}$. For the waves propagating in the xy plane, these equations are factored into three modes whose vibration directions are mutually perpendicular to each other: shear-horizontally (SH) polarized pure transverse (PT) mode vibrating in the z direction, quasilongitudinal (QL) mode and quasitransverse (QT) mode. The QL and QT modes are both polarized in the xy plane. The SH polarized PT mode is uncoupled from the QL and QT modes. We first deal with the QL and QT modes.

Quasilongitudinal and Quasitransverse Modes

For simplicity of notation we introduce the following identities: $C_{11\pm} = C_{11} \pm C_{66}$, $C_{22\pm} = C_{22} \pm C_{66}$, and $C_{12\pm} = C_{12} \pm C_{66}$. The phase velocities of the QL and QT modes are given by [5]

$$2\rho v^2 = C_{11+}\cos^2\alpha + C_{22+}\sin^2\alpha \pm \sqrt{D}, \tag{15}$$

where α is defined in Eq. (14), the upper and lower signs in $\pm$ in front of $\sqrt{D}$ correspond to the QL and QT modes, respectively, and

$$D = \left(C_{11-}\cos^2\alpha - C_{22-}\sin^2\alpha\right)^2 + 4C_{12+}^2\sin^2\alpha\cos^2\alpha. \tag{16}$$

Differentiating Eq. (15) with respect to θ, one obtains

$$\frac{v'}{v} = \frac{1}{4\rho v^2}\left\{\begin{array}{l}(C_{22+} - C_{11+})\sin 2\alpha \\ \pm\dfrac{1}{\sqrt{D}}\left[\left(C_{22-}\sin^2\alpha - C_{11-}\cos^2\alpha\right)\left(C_{11-} + C_{22-}\right)\sin 2\alpha + C_{12+}^2\sin 4\alpha\right]^{1/2}\end{array}\right\}. \tag{17}$$

Substitution of the Snell's law Eq. (9) into Eq. (15) yields

$$2\rho\sin^2\theta = h^2\left[C_{11+}\cos^2\left(\theta - \beta_f\right) + C_{22+}\sin^2\left(\theta - \beta_f\right) \pm \sqrt{D}\right]. \tag{18}$$

where $h = \sin\theta / v$ is the Snell's law constant. Consider a case in which rays are initiating from a broadband source in virtually every direction inside the specimen. We choose an arbitrary ray whose wave normal at the source is directed at an angle θ_o from the x direction. The fiber direction β_f at the source is obtained from Eq. (13). Then, we calculate the initial velocity v_o corresponding to the wave normal θ_o via Eq. (15) to determine the Snell's law constant h from Eq. (9). For a given x, the fiber angle β_f can be calculated via Eq. (13) and from Eq. (18), one obtains the solutions for θ, $\sin\theta$, and $\cos\theta$. Then the values of v and v' are obtained from Eqs. (15) and (17). Thus, one can calculate the values of $\sin\theta$, $\cos\theta$, and v and v' at many different values of x to determine the arrival time τ and ray path through Eqs. (11) and (12). One can repeat this procedure for rays with various initial wave normal directions and at different source points.

Pure Transverse Mode

The phase velocity v of the PT mode is given by [5]

$$\rho v^2 = C_{44}\sin^2\left(\theta - \beta_f\right) + C_{55}\cos^2\left(\theta - \beta_f\right). \tag{19}$$

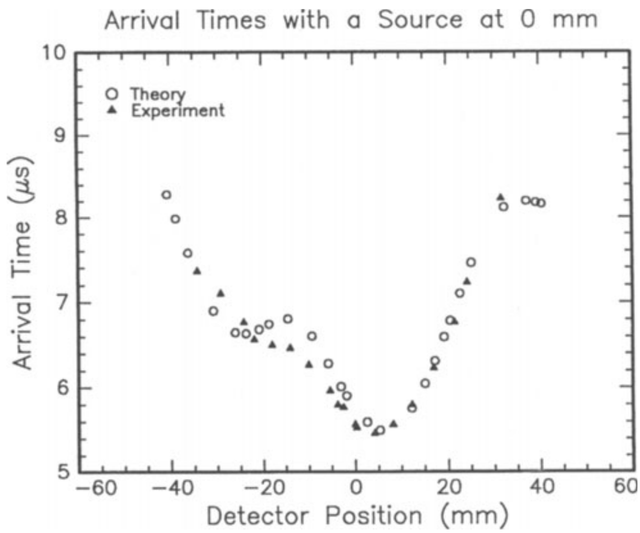

Figure 2. Comparison of the arrival times of the QL mode between theory and experiment.

Determination of the arrival time and ray path of a wave with initial wave normal θ_o at the source can be carried out in a similar way to that described in the case of the QL and QT modes. In the special case of $h = \sin\theta = 0$, the integrals for the arrival time and ray path are greatly simplified for all three modes.

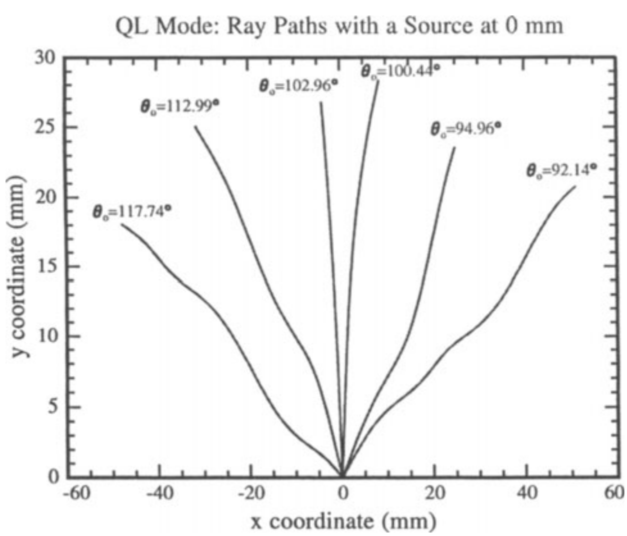

Figure 3. Computed ray paths of the QL mode with a source at $x = 0$ mm.

RESULTS

The fabricated wavy composites are flatly machined and polished with two opposite flat faces parallel to the mean fiber direction, x axis. A pointlike source is generated by breaking a glass capillary of 0.1 mm diameter with a razor blade on the 28 μm thick, piezoelectric polyvinylidene fluoride (PVDF) film, which was mounted on the flat surface of the wavy composite. The generated elastic waves propagate in virtually every direction inside the specimen and are detected by a small piezoelectric transducer on the flat side opposite to the source's. The distance between the flat surfaces is 15.93 mm. Figure 2 shows comparison of the arrival time data of the QL mode between those calculated according to Eq. (11) and those measured, when the source is located at origin and the elastic waves are detected at various positions. Good agreement between theory and experiment is found in the figure. The theoretical ray paths corresponding to the above experimental configuration are calculated via Eq. (12) and are displayed in Figure 3. For other combinations of source/detector positions refer to Ref. 6.

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