Equivalence of D_{ijkl} to Q_{ijkl}

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Abstract

The author of this article introduced two important physical variables, K_{ijkl} in Eq. 10 of Ref. 1 and D_{ijkl}, which defines the ultrasonic equation of motion as

$D_{ijkl} \equiv \frac{1}{2}(B_{ijkl} + B_{iljk}) = \sigma_{kl}\delta_{ij} + \frac{1}{2}(C_{ikjl} + C_{iljk})$, where

$B_{ijkl}(\mathbf{X}) = (\frac{\rho_X \, \partial^2 U}{\partial(x_k/X_l)\partial(x_l/X_k)})_{\boldsymbol{S};\boldsymbol{X}} = \delta_{ik}\sigma_{jl} + C_{ijkl}(X)$, in Eq. (19-22) of Ref. 2,

$\rho_X \ddot{u}_i = D_{ijkl} \frac{\partial^2 u_j}{\partial X_k \partial X_l}$ in Eq. 70 of Ref. 1.

This article shows that D_{ijkl} is equivalent to K_{ijkl} .

Keywords: Equivalence of D_{ijkl} to K_{ijkl}

I. Introduction

At first, we introduce three deformation states characterized as indices a, X, and i, where index a represents an undeformed stress-free state, index X characterizes a static finite deformation state from the undeformed state a, and index i represents a small deformation state superposed on the finite deformation state X by a travelling ultrasonic wave. Two variables α_{ij} and β_{ij} are defined as

$$\alpha_{ij} \equiv {\partial x_i}/{\partial X_j} \qquad\qquad \beta_{ij} \equiv {\partial X_i}/{\partial x_j}\,. \tag{1}$$

Introduce now displacement gradient u_{ij} defined as

$$u_{ij} \equiv {\partial u_i}/{\partial X_j}. \tag{2}$$

Then,

$$\epsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji}) = \frac{1}{2}(\alpha_{ij} + \alpha_{ji} - 2\delta_{ij})\,, \tag{3}$$

We also define in the initial state **X** the effective elastic stiffness K_{ijkl} and the effective elastic compliance Q_{ijkl} of the second order, which indicate the measure of material strength in the initial state, as [see Ref. 1]

$$K_{ijkl} = (\frac{\partial\sigma_{ij}}{\partial\epsilon_{kl}})_{\boldsymbol{X}} = C_{ijkl}(X) - \sigma_{ij}(X)\delta_{kl} + \frac{1}{2}\,(\delta_{jk}\sigma_{il}(\mathrm{X}) + \delta_{il}\sigma_{jk}(\mathrm{X}) + \delta_{ik}\sigma_{jl}(\mathrm{X})), \tag{4}$$

$$[K_{\alpha\beta}] = \begin{bmatrix} C_{11}+\sigma_1 & C_{12}-\sigma_1 & C_{13}-\sigma_1 & C_{14} & C_{15}+\sigma_5 & C_{16}+\sigma_6 \\ C_{11}-\sigma_2 & C_{22}+\sigma_2 & C_{23}-\sigma_2 & C_{24}+\sigma_4 & C_{25} & C_{26}+\sigma_6 \\ C_{13}-\sigma_3 & C_{23}-\sigma_3 & C_{33}+\sigma_3 & C_{34}+\sigma_4 & C_{35}+\sigma_5 & C_{36} \\ C_{14}-\sigma_4 & C_{24} & C_{34} & C_{44}+(\sigma_2+\sigma_3)/2 & C_{45}+\sigma_6/2 & C_{46}+\sigma_5/2 \\ C_{15} & C_{11}+\sigma & C_{35} & C_{45}+\sigma_6/2 & C_{44}+(\sigma_2+\sigma_3)/2 & C_{56}+\sigma_4/2 \\ C_{16} & C_{26} & C_{36}-\sigma_6 & C_{46}+\sigma_5/2 & C_{56}+\sigma_4/2 & C_{44}+(\sigma_2+\sigma_3)/2 \end{bmatrix} \tag{5}$$

In the above Eqs. 4 and 5, K_{ijkl} or $[K_{\alpha\beta}]$ can be conveniently obtained by measuring the Cauchy stresses σ_i (i = 1,2   6), measurements of which in uniaxial loading case are described in detail [see Ref. 3], and the second-order elastic stiffness constants C_{ij} (i, j = 1,2,   6), which can be accurately obtained by ultrasonic wave-speeds measurements [see Ref. 2].

It is noted that full symmetry relations as found in C_{ijkl} are lost in K_{ijkl} and Q_{ijkl}. K_{ijkl} obey the relations $K_{ijkl} = K_{klij} = K_{ijlk}$ along with

$$K_{ijkl} - K_{klij} = \delta_{ij}\sigma_{kl} - \delta_{kl}\sigma_{ij} \tag{6}$$

$$K_{ijkl}Q_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \qquad\qquad K_{\alpha\beta}Q_{\beta\gamma} = \delta_{\alpha\gamma}. \tag{7}$$

Here isothermal compressibility $x^T(\mathbf{X})$ is introduced as

$$x^T(\mathbf{X})\ (\text{Compressibility}) = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = -\left(\frac{\partial J}{\partial P}\right)_T, \tag{8}$$

where p and J denote the hydrostatic pressure and the Jacobian, respectively.

$$x^T(\mathbf{X}) = -\left(\frac{\partial J}{\partial P}\right)_{T;\mathbf{X}} = Q_{iikk}(\mathbf{X})$$

In a stressed state Ref. 2 defines in Eq. (19.22) the isentropic elastic stiffness coefficients $B_{ijkl}(\mathbf{X})$ **as**

$$B_{ijkl}(\mathbf{X}) = \left(\frac{\rho_X \, \partial^2 U}{\partial(x_k/X_l)\partial(x_l/X_k)}\right)_{S;X} = \delta_{ik}\sigma_{jl} + C_{ijkl}(X), \tag{9}$$

where U is the internal energy of deformation of a material. Note that

$$B_{ijkl} = T_{jl}\,\delta_{ik} + C_{ijkl}; \quad B_{klij} = B_{ijkl}; \quad B_{ijkl} \neq B_{jikl}; \quad B_{ijlk} \neq B_{ijkl} \ . \tag{10}$$

At $\mathbf{X}$ let $D_{ijkl} \equiv \frac{1}{2}(B_{ijkl} + B_{iljk}) = \sigma_{kl}\delta_{ij} + \frac{1}{2}(C_{ikjl} + C_{iljk})$. (11)

Then,

$$D_{ijkl}(\mathbf{X}) - D_{klij}(\mathbf{X}) = K_{ijkl}(\mathbf{X}) - K_{klij}(\mathbf{X}) = \delta_{ij}\sigma_{kl} - \delta_{kl}\sigma_{ij}, \tag{12}$$

which is known as Huang's conditions.

Ultrasonic equation of motion at $\mathbf{X}$ is expressed as

$$\rho_X\, \ddot{u}_i = D_{ijkl}\frac{\partial^2 u_j}{\partial X_k \partial X_l} \tag{13}$$

II. Relation between K_{ijkl} and D_{ijkl}

$$D_{ijkl} = \delta_{ij}\sigma_{kl} + (1/2)\,(C_{ikjl} + C_{iljk}) \tag{14}$$

$$D_{ikjl} = \delta_{ik}\sigma_{jl} + (1/2)\,(C_{ijkl} + C_{iljk}) \tag{15}$$

$$D_{ijkl} = \delta_{il}\sigma_{jk} + (1/2)\,(C_{ijkl} + C_{ikjl}) \tag{16}$$

Eq. 15 minus Eq. 14 plus Eq. 16 results in

$$D_{ikjl} - D_{ijkl} + D_{iljk} = \delta_{ik}\sigma_{jl} - \delta_{ij}\sigma_{kl} + \delta_{il}\sigma_{jk} + C_{ijkl} \tag{17}$$

Therefore, $C_{ijkl} = D_{ikjl} - D_{ijkl} + D_{iljk} + \delta_{ij}\sigma_{kl} - \delta_{ik}\sigma_{jl} + \delta_{il}\sigma_{jk}$ (18)

$$K_{ijkl} = C_{ijkl} - \delta_{ij}\sigma_{kl} + (1/2)(\delta_{ik}\sigma_{jl} + \delta_{jk}\sigma_{il} + \delta_{il}\sigma_{jk} + \delta_{ik}\sigma_{jl})$$

$$= \frac{1}{2}(D_{ikjl} + D_{iljk} + D_{jkil} + D_{jlik}) - D_{klij} \tag{19}$$

III. Equivalence of D_{ijkl} to K_{ijkl}

Now we consider four cases of ijkl indices of D.

Case 1. Three indices of i,j,k,&l are equal, i.e., i=j=k or i=j=l or i=k=l.

Then, $K_{ijkl} = D_{ijkl}$

Case 2. Two indices are equal. i.e., i=j or k=l, but i≠ k

Then, $K_{ijkl} = 2D_{ikjl} - D_{klij}$

Case 3. Two indices are equal, i.e., (i=k & j=l) or (i=l & j=k), but (i≠ j & k ≠ l)

Then, $K_{ijkl} = (D_{ikjl} + D_{jlik})/2$

Case 4. ((i≠ j & k ≠ l) & i = k & j ≠ l or (i≠ k & j = l)) or (i = l & j ≠ k) or (i≠ l & j = k)

Then, $K_{ijkl} = (D_{iljk} + D_{jkil})/2$

In all cases either $K_{ijkl} = D_{ijkl}$ or K_{ijkl} is a linear combination of two D's coefficients.

IV. Conclusion

This work demonstrates the equivalence of D_{ijkl} to K_{ijkl}, which indicates the measure of material strength in the initial state X.

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