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Group velocity formulas for the symmetry planes of a stressed anisotropic elastic solid

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This paper presents analytic formulas for the group velocity of quasilongitudinal, quasitransverse, and shear-horizontally (SH) polarized pure-transverse modes propagating in an arbitrary direction on the symmetry planes of a stressed anisotropic elastic medium with orthotropic or higher symmetry. The group velocity equations are expressed in terms of the thermodynamic elastic stiffness coefficients and stresses acting on the medium. An example is provided with a (001) silicon crystal compressed at uniaxial stress.
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INTRODUCTION

The group velocities of various modes of an elastic wave propagating in an elastic anisotropic medium in the stress-free natural state have been extensively treated by many authors.[1–3] Explicit analytic formulas for phase velocities in the symmetry direction and in an arbitrary direction of symmetry planes are also given in the literature.[1,2,4] Because of the complexity of the group velocity surfaces in an anisotropic medium, no explicit analytic formula for the group velocity in a general direction exists. However, in the symmetry directions, the phase and group velocities coincide and this leads to valuable relations between the group velocity and elastic constants of the medium. Recently, the first author extended the group velocity expressions to an arbitrary direction on the symmetry planes[5] and Kim *et al.*[6–9] gave the detailed treatment on the methods of determining all the elastic constants of an anisotropic medium from group velocity data measured in symmetry directions and planes. Based on the two-dimensional Stroh formalism for the elastodynamic problems, Wang[10] gave an elegant treatment for the cusps of the group velocity surfaces.

Equations of phase velocities at finite deformation of an elastic medium under arbitrary stresses were formulated by Toupin and Bernstein[11] and Thurston.[12,13] In the symmetry directions of a stressed medium that maintains orthotropic or higher symmetry, the group and phase velocities coincide with each other, as in the case of an stress-free medium. This gives relations between the group velocities of the pure mode to the diagonal elements of the elastic constant matrix. However, to the authors' knowledge, there appears no explicit analytic formulation for the group velocity in an arbitrary direction of symmetry planes. In this paper we derive the group velocity expressions for the shear-horizontally (SH) polarized pure-transverse (PT) modes, quasilongitudinal (QL) and quasitransverse (QT) modes propagating on the symmetry planes of the stressed medium. Our approach is basically an extension of the methods used in Ref. 5 to the stressed medium, replacing Christoffel's tensor by the equivalent acoustical tensor in the stressed state, where the group velocity direction is again found to be normal to the equivalent slowness surface in the stressed state.

The elastic waves emanating from their sources propagate at the speed of group velocities which depend on the direction of propagation in an anisotropic medium. Since the group velocity, just like the phase velocity, also depends on the stress on the medium, the measurement of group velocity may yield information about the stresses acting on the medium. This effect, known as acoustoelasticity,[14] is generally small in the moderate stress range below 1 GPa, and still detectable if one measures the wave speed very accurately. However, the change in group or phase velocity will be significant in very high stresses, which can be found in the interior of planets such as the Earth and Jupiter and inside the diamond-anvil high-pressure cell in the laboratory,[15] where the stresses acting on a material may be much higher than its Young's modulus in the natural state. A study of group velocity will contribute to the understanding of the acoustoelastic effect and the behavior of a material under high pressures.

I. PHASE AND GROUP VELOCITIES, AND SLOWNESS OF A STRESSED MEDIUM: GENERAL FORMULATION

Suppose that a small amplitude wave motion $\mathbf{u}$ is superposed on the finite deformation caused by static stresses $\sigma_{ij}(\mathbf{X})$ acting on the medium. We denote the coordinate of a particle of a stressed elastic body at finite deformation state by $\mathbf{X}$, which we adopt as a reference coordinate for deformation. The equation of motion for the deformation **u** of a homogeneous medium is written in the absence of body force as[11,12]

$$\rho_X \ddot{u}_i = [\delta_{ik}\sigma_{jl}(\mathbf{X}) + C_{ijkl}(\mathbf{X})] \frac{\partial^2 u_k}{\partial X_j \, \partial X_l} = B_{ijkl} \frac{\partial^2 u_k}{\partial X_j \, \partial X_l}, \tag{1}$$

where ρ_X is the material density at $\mathbf{X}$ and $B_{ijkl} \equiv \delta_{ik}\sigma_{jl}(\mathbf{X}) + C_{ijkl}(\mathbf{X})$. $C_{ijkl}(\mathbf{X})$, the thermodynamic elastic coefficient evaluated at $\mathbf{X}$, is defined at constant entropy S as

$$C_{ijkl}(\mathbf{X}) = \rho_X \left(\frac{\partial^2 U}{\partial \xi_{ij} \, \partial \xi_{kl}} \right)_{S;\mathbf{X}}. \tag{2}$$

In Eq. (2), U is the internal energy per unit mass of the material, the strain from the reference state $\mathbf{X}$ is given by

$$\xi_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_s}{\partial X_i} \frac{\partial u_s}{\partial X_j} \right), \tag{3}$$

and the thermodynamic coefficients $C_{ijkl}(\mathbf{X})$ have a familiar symmetry as the elastic stiffness constants defined in the stress-free natural state do.

However, the elastic coefficients B_{ijkl} lack the full symmetry found in $C_{ijkl}(\mathbf{X})$ and cannot be expressed using the abbreviated Voigt notation. We rearrange B_{ijkl} and define a new set of wave propagation coefficients $\tilde{C}_{ijkl}$, which can be abbreviated using Voigt's notation. Following Huang[16] and Born and Huang,[17] we write $\tilde{C}_{ijkl}$ as

$$\tilde{C}_{ijkl} = (B_{ikjl} + B_{iljk})/2 = \delta_{ij}\sigma_{kl} + (C_{ikjl} + C_{iljk})/2. \tag{4}$$

Note that $\tilde{C}_{ijkl} = \tilde{C}_{jikl}$ and $\tilde{C}_{ijkl} = \tilde{C}_{ijlk}$. $\tilde{C}_{ijkl}$ obeys Huang's condition

$$\tilde{C}_{ijkl} - \tilde{C}_{klij} = \tilde{C}_{\mu\nu} - \tilde{C}_{\nu\mu} = \delta_{ij}\sigma_{kl} - \delta_{kl}\sigma_{ij}, \tag{5}$$

where the subscripts μ and ν ($\mu, \nu = 1,2,\ldots,6$) are the Voigt indices. The 6×6 array $\tilde{C}_{\mu\nu}$ is shown in Table III of Ref. 12 and has in general 26 linearly independent elements. Using the $\tilde{C}_{ijkl}$ coefficients, the equation of motion is expressed as

$$\rho_X \ddot{u}_i = \tilde{C}_{ijkl} \frac{\partial^2 u_j}{\partial X_k \, \partial X_l}. \tag{6}$$

Writing the acoustical tensor as

$$\Gamma_{ij}(\mathbf{n}) = \tilde{C}_{ijkl} n_k n_l \tag{7}$$

for the plane wave propagating with wave vector $\mathbf{k} = 2\pi\mathbf{n}/\lambda$, wave length λ, wave normal $\mathbf{n}$, and phase velocity $\mathbf{V}$, one obtains the phase velocity equations

$$(\tilde{C}_{ijkl} n_k n_l - \rho_X V^2 \delta_{ij}) u_j = 0, \tag{8}$$

$$\det|\Gamma_{ij} - \rho_X V^2 \delta_{ij}| = 0. \tag{9}$$

Γ_{ij} is the symmetrical tensor whose eigenvectors are the possible directions of particle displacement and whose eigenvalues are the corresponding values of $\rho_X V^2$.

In terms of slowness, $\mathbf{s} = \mathbf{n}/V = \mathbf{k}/\omega$ defined as the inverse phase velocity, where ω denotes the angular frequency, the slowness surface Λ of the stressed medium is represented by

$$\Lambda = \det|\tilde{C}_{ijkl} s_k s_l - \rho_X \delta_{ij}| = 0. \tag{10}$$

The group velocity $\mathbf{V}_g$, commonly defined as

$$\mathbf{V}_g \equiv \boldsymbol{\nabla}_k \omega, \tag{11}$$

satisfies the relations

$$\mathbf{V}_g \cdot \mathbf{n} = V, \quad \mathbf{V}_g \cdot \mathbf{s} = 1. \tag{12}$$

From Eqs. (10) and (12) it can be shown that[1]

$$\mathbf{V}_g = \frac{\boldsymbol{\nabla}_s \Lambda}{\mathbf{s} \cdot \boldsymbol{\nabla}_s \Lambda}, \tag{13}$$

which indicates that the group velocity points in the direction normal to the slowness surface, as in the case of the stress-free natural state. Note that Eq. (13) holds valid for a stressed medium as a result of $\Gamma_{ij} = \Gamma_{ji}$ in Eq. (7), $\tilde{C}_{ijkl} \neq \tilde{C}_{klij}$ notwithstanding.

In the following we will restrict ourselves to the wave propagation with wave normal **n** lying in the symmetry plane of a medium possessing orthotropic or higher symmetry, where the three axes of orthotropic symmetry are taken as the coordinate axes, X_1, X_2, and X_3, whose directions are simply denoted as [100], [010], and [001], respectively. Likewise, we denote the X_1X_2 plane whose normal points in the X_3 direction by (001), and analogously for the X_2X_3 and X_1X_3 planes. The orthotropic medium is characterized by nine thermodynamic, elastic-stiffness coefficients: C_{11}, C_{22}, C_{33}, C_{12}, C_{23}, C_{13}, C_{44}, C_{55}, and C_{66}, just as an orthorhombic medium in the stress-free natural state is. Here, we deal with only those media which possess three mutually perpendicular symmetry planes, and therefore exclude materials of triclinic, monoclinic, and trigonal symmetries. A medium, which has orthorhombic, tetragonal (422, $4mm$, $\bar{4}2m$, and $4/mmm$ classes), cubic, hexagonal (622, $6mm$, $\bar{6}2m$, and $6/mmm$ classes), transversely isotropic, or isotropic symmetry in the stress-free natural state, can be considered as a medium with orthotropic or higher symmetry, when it is uniaxially, biaxially, or triaxially loaded with the directions of the principal-stress axes coinciding with those of material symmetry. This condition for maintenance of orthotropic or higher symmetry can be stated as

$$\begin{aligned} &\sigma_{12} = \sigma_{13} = \sigma_{23} = 0 \quad \text{or} \\ &\partial X_i / \partial a_j = \lambda_i \delta_{ij} \quad (i \ \text{not summed}), \end{aligned} \tag{14}$$

where a_j represents a coordinate of a particle along the j-th direction in the stress-free natural state and λ_i is a principal stretch in the i-th direction. Note that the symmetry relation $C_{\mu\nu} = C_{\nu\mu}$ holds for the medium in the natural state, while Huang's condition Eq. (5) holds for a stressed medium.

II. PHASE VELOCITIES OF A STRESSED MEDIUM

Phase velocities in a stressed anisotropic medium are treated in detail in Refs. 12–14. We introduce this section as a reference that is necessary for the derivation of group velocities to be presented in Sec. III. For waves propagating on symmetry planes of an elastic medium, we choose specifically, without loss of generality, a wave normal $\mathbf{n} = [n_1, 0, n_3] = [\sin\theta, 0, \cos\theta]$ lying on the (010) plane at an angle θ to the [001] direction. Because of the mirror symmetry across the (001) plane, we restrict the angle θ to $-90° \leqslant \theta \leqslant 90°$. Wave propagations in the (100) and (001) symmetry planes can be treated by the proper rotation of indices.

The acoustical tensors Γ_{ij} in the (010) plane can be found in Table III of Ref. 12 by setting $n_2=0$ and $\sigma_{13}=0$.

Equation (9) for the wave propagation in the (010) plane yields

$$(\Gamma_{22}-\rho_X V^2)[(\Gamma_{11}-\rho_X V^2)(\Gamma_{33}-\rho_X V^2)-\Gamma_{13}^2]=0. \quad (15)$$

For simplicity of notation we introduce the following identities:

$$C_{11\pm}\equiv C_{11}\pm C_{55}, \quad C_{33\pm}\equiv C_{33}\pm C_{55}, \quad C_{13\pm}\equiv C_{13}\pm C_{55}; \quad (16)$$

for the pure-index, effective elastic coefficients $C_{\mu\mu}^{(i)}$ (μ not summed; $\mu=1,2,...,6$),

$$C_{\mu\mu}^{(i)}\equiv C_{\mu\mu}+\sigma_{ii} \quad (i \text{ not summed}; i=1,2,3); \quad (17)$$

for mixed-index, effective elastic coefficients $C_{\mu\nu}^{(i)}$ ($\mu\neq\nu$; $\mu,\nu=1,2,3$),

$$C_{\mu\nu}^{(i)}\equiv C_{\mu\nu}-\sigma_{ii} \quad (i \text{ not summed}; i=1,2,3); \quad (18)$$

and for the following effective elastic-stiffness constants

$$C_{11\pm}^{(1)}\equiv C_{11}^{(1)}\pm C_{55}^{(1)}, \quad C_{13\pm}^{(1)}\equiv C_{13}^{(1)}\pm C_{55}^{(1)}, \quad C_{33\pm}^{(3)}\equiv C_{33}^{(3)}\pm C_{55}^{(3)}, \quad C_{13\pm}^{(3)}\equiv C_{13}^{(3)}\pm C_{55}^{(3)}. \quad (19)$$

The first term in the parenthesis of Eq. (15) represents the pure-transverse (PT) mode polarized in the [010] direction and propagating with phase velocity

$$\rho_X V^2=\Gamma_{22}=C_{66}^{(1)}\sin^2\theta+C_{44}^{(3)}\cos^2\theta \quad (PT \text{ mode}). \quad (20)$$

The square bracket term in Eq. (15) yields the phase velocities for the quasilongitudinal (QL) and quasitransverse (QT) modes propagating on the (010) plane and polarized on the same plane:

$$(\rho_X V^2)^2-(\Gamma_{11}+\Gamma_{33})(\rho_X V^2)+(\Gamma_{11}\Gamma_{33}-\Gamma_{13}^2)=0, \quad (21)$$

$$\begin{aligned}2\rho_X V^2&=\Gamma_{11}+\Gamma_{33}\pm[(\Gamma_{11}-\Gamma_{33})^2+4\Gamma_{13}^2]^{1/2}\\&=C_{11+}^{(1)}\sin^2\theta+C_{33+}^{(3)}\cos^2\theta\\&\quad\pm[(C_{11-}\sin^2\theta-C_{33-}\cos^2\theta)^2\\&\quad+4C_{13+}^2\sin^2\theta\cos^2\theta]^{1/2},\end{aligned} \quad (22)$$

where + and − signs in front of the square bracket refer to the QL and QT modes, respectively.

Equations (20) and (22) express the phase velocities of various modes propagating in an arbitrary direction on the (010) symmetry plane of a stressed medium. Similar expressions can be found for the other symmetry planes, (100) and (001), by an appropriate rotation of indices for the elastic constants and stresses. In particular, for the pure-longitudinal and pure-transverse modes propagating in the symmetry directions, one can easily find in matrix form

$$\rho_X[V_{ij}^2]=\begin{bmatrix}C_{11}^{(1)} & C_{66}^{(1)} & C_{55}^{(1)}\\ C_{66}^{(2)} & C_{22}^{(2)} & C_{44}^{(2)}\\ C_{55}^{(3)} & C_{44}^{(3)} & C_{33}^{(3)}\end{bmatrix}, \quad (23)$$

where V_{ij} denotes the phase velocity propagating in the X_i direction and polarized in the X_j direction. Equation (23) indicates that all the pure-index or diagonal-element thermodynamic elastic coefficients can be determined from measurements of the pure-mode phase velocities propagating in three symmetry directions. Note that the elastic constants C_{11-} and C_{33-} appearing in Eq. (22) can be similarly determined using Eq. (23), since $C_{11-}=C_{11-}^{(1)}$ and $C_{33-}=C_{33-}^{(3)}$. This means that the elastic coefficient $C_{13+}=C_{13}+C_{55}$ appearing in Eq. (22) can also be determined from measurements of the QL- or QT-mode phase velocity together with the pure-mode phase velocities propagating in symmetry directions. Note also that $C_{13+}=C_{13+}^{(1)}=C_{13+}^{(3)}$ and therefore, $C_{13}^{(1)}$ and $C_{13}^{(3)}$ in Eq. (18), which appear in the formulas of the effective Young's modulus and Poisson's ratios of a stressed orthotropic medium,[18,19] can also be similarly obtained from measurements of relevant pure- and QL- (or QT-) mode phase velocities.

Since the phase and group velocities are identical for waves propagating along the symmetry directions, Eq. (23) also applies for the pure-mode group velocity by simply replacing V_{ij} by $(V_g)_{ij}$. On the other hand, along an off-symmetry direction on the symmetry plane, the direction of the group velocity deviates from that of the wave normal, and in the following section we deal with the derivation of the group velocity formulas and their application to determination of the group velocity surfaces and the mixed-index elastic coefficients.

III. GROUP VELOCITIES OF A STRESSED MEDIUM

The group velocity corresponding to a wave normal $\mathbf{n}=[\sin\theta,0,\cos\theta]$ in the (010) slowness plane can be calculated using Eq. (13). Because of the mirror symmetry across the (010) symmetry plane, all the points in the (010) slowness plane map onto the (010) plane of the group-velocity surface. However, except for isotropic and transversely isotropic media, the converse is not generally true, as is well known in the theory of phonon focusing.[20,21] Because of the nonspherical, concave, or convex shape of the QT slowness surface of an anisotropic medium, some points that do not lie in the (010) section of the QT slowness surface may map onto the (010) group-velocity section. The group-velocity sections that do not correspond to the (010) slowness plane are not of interest here. Hence, we deal with only those group velocity sections that correspond to the (010) slowness plane. We denote the direction of group velocity by an angle ζ measured to the [001] direction. Because of the mirror symmetry across the (001) plane, we confine ζ to $-90° \leqslant \zeta \leqslant 90°$, just as θ. For simplicity of notation we write

$$p\equiv\tan\theta, \quad q\equiv\tan\zeta. \quad (24)$$

A. Pure-transverse mode

Equation (20) yields the equation of the (010) slowness section of the pure-transverse mode

$$\Lambda_{PT}=C_{66}^{(1)}s_1^2+C_{44}^{(3)}s_3^2-\rho_X=0, \tag{25}$$

$$\rho_X s_3^{-2}=C_{66}^{(1)}p^2+C_{44}^{(3)}. \tag{26}$$

Applying Eq. (13) to Eqs. (25) and (26), one obtains

$$V_{g1}=s_3 C_{66}^{(1)}p/\rho_X, \tag{27a}$$

$$V_{g3}=s_3 C_{44}^{(1)}/\rho_X, \tag{27b}$$

$$q=\tan\,\zeta=\frac{V_{g1}}{V_{g3}}=\frac{C_{66}^{(1)}p}{C_{44}^{(3)}}, \tag{28a}$$

$$p=\tan\,\theta=\frac{C_{44}^{(3)}q}{C_{66}^{(1)}}, \tag{28b}$$

$$(\rho_X V_g^2)^{-1}=[\rho_X(V_{g1}^2+V_{g3}^2)]^{-1} =(C_{66}^{(1)})^{-1}\sin^2\,\zeta+(C_{44}^{(3)})^{-1}\cos^2\,\zeta. \tag{29}$$

Equations (28a) and (28b) give the conversion relations between the directions of phase and group velocities of the *SH*-polarized PT mode. Equation (29) indicates the elliptical section of V_g with the principal semiaxes given by $\sqrt{C_{44}^{(3)}/\rho_X}$ and $\sqrt{C_{66}^{(1)}/\rho_X}$.

B. Quasilongitudinal and quasitransverse modes

We again introduce for simplicity of notation

$$A\equiv C_{11}^{(1)}C_{33}^{(3)}+C_{55}^{(1)}C_{55}^{(3)}-C_{13+}^2, \tag{30}$$

$$B\equiv C_{11-}C_{33-}-2C_{13+}^2=C_{11-}^{(1)}C_{33-}^{(3)}-2C_{13+}^{(1)}C_{13+}^{(3)}, \tag{31}$$

$$D\equiv\frac{1}{n_3^2}[(\Gamma_{11}-\Gamma_{33})^2+4\Gamma_{13}^2]^{1/2} =[(C_{11-}p^2-C_{33-})^2+4C_{13+}^2p^2]^{1/2}, \tag{32}$$

$$F\equiv C_{11}^{(1)}C_{55}^{(1)}s_1^4+C_{33}^{(3)}C_{55}^{(3)}s_3^4+As_1^2s_3^2, \tag{33}$$

$$G\equiv\rho_X(C_{11+}^{(1)}s_1^2+C_{33+}^{(3)}s_3^2), \tag{34}$$

$$U_1\equiv 2C_{11}^{(1)}C_{55}^{(1)}p^2+A-\rho_X s_3^{-2}C_{11+}^{(1)}, \tag{35}$$

$$U_3\equiv 2C_{33}^{(3)}C_{55}^{(3)}+Ap^2-\rho_X s_3^{-2}C_{33+}^{(3)}, \tag{36}$$

$$Q\equiv C_{11+}^{(1)}p^2+C_{33+}^{(3)}-2\rho_X s_3^{-2}, \tag{37}$$

where the quantity $\rho_X s_3^{-2}$ can be obtained from Eq. (22) and expressed as

$$2\rho_X s_3^{-2}=C_{11+}^{(1)}p^2+C_{33+}^{(3)}\pm D. \tag{38}$$

The positive and negative signs in front of D in Eq. (38) correspond to the *QL* and *QT* modes, respectively. D is by definition always greater than zero in an anisotropic medium. Substitution of Eq. (38) into Eq. (37) yields the identity

$$Q=\mp D, \tag{39}$$

where the negative and positive signs correspond to the QL and QT modes, respectively. B in Eq. (31) and D in Eq. (32) are related by

$$B=\frac{1}{2p^2}(C_{11-}^2p^4+C_{33-}^2-D^2). \tag{40}$$

The group velocities of both the QL and QT modes can be found analytically from the equation of the slowness surface, which can be derived from Eq. (21) as

$$\Lambda=F-G+\rho_X^2=0, \tag{41}$$

where F and G, specified by Eqs. (33) and (34), are respectively homogeneous functions of degree 4 and 2 in s. Using Euler's theorem on a homogeneous function, it is easy to show that

$$s\cdot\nabla_s\Lambda=4F-2G=2(G-2\rho_X^2)=2\rho_X s_3^2 Q. \tag{42}$$

From Eq. (13) one obtains

$$V_{g1}=\frac{s_1U_1}{\rho_XQ},\quad V_{g3}=\frac{s_3U_3}{\rho_XQ}, \tag{43}$$

$$q\equiv\tan\,\zeta=\frac{V_{g1}}{V_{g3}}=\frac{s_1U_1}{s_3U_3}=\frac{U_1p}{U_3}, \tag{44}$$

$$\rho_XV_g^2=\rho_X(V_{g1}^2+V_{g3}^2)=\frac{(1+q^2)U_3^2}{\rho_Xs_3^{-2}Q^2}. \tag{45}$$

To proceed further, we first consider the QL mode and then the QT mode.

1. Quasilongitudinal mode

For the QL-mode propagation the upper sign in front of Eqs. (38) and (39) applies to Eqs. (35)–(37), (44), and (45). After some algebra, U_1 and U_3 in Eq. (44) reduce to

$$U_1=(B-C_{11-}^2p^2-C_{11+}^{(1)}D)/2, \tag{46}$$

$$U_3=(Bp^2-C_{33-}^2-C_{33+}^{(3)}D)/2. \tag{47}$$

Substitution of Eqs. (46) and (47) into Eq. (44) leads to the following relationship

$$q=\frac{p(B-C_{11-}^2p^2-C_{11+}^{(1)}D)}{Bp^2-C_{33-}^2-C_{33+}^{(3)}D}, \tag{48}$$

which can also be expressed in the form of

$$C_{11-}^2p^3+q(Bp^2-C_{33-}^2)-Bp+(C_{11+}^{(1)}p-C_{33+}^{(3)}q)D=0. \tag{49}$$

Equation (48) or (49) can be used to find the direction of a wave normal p corresponding to that of a group velocity lying in the (010) plane and vice versa, when the relevant values of thermodynamic elastic coefficients and stresses exerted in a medium are known. Substituting Eq. (40) into Eq. (49) and rearranging the resulting equation in terms of powers in D, one obtains

$$(1-pq)D^2-2p(C_{33+}^{(3)}q-C_{11+}^{(1)}p)D+(1+pq) \times(C_{11-}^2p^4-C_{33-}^2)=0, \tag{50}$$

which yields

$$D=\frac{1}{1-pq}\{p(C_{33+}^{(3)}q-C_{11+}^{(1)}p)\pm[p^2(C_{33+}^{(3)}q-C_{11+}^{(1)}p)^2 -(1-p^2q^2)(C_{11-}^2p^4-C_{33-}^2)]^{1/2}\}. \quad (51)$$

For a given group direction q in the above equation, we choose the region of p which makes D real and positive. Finally, from Eqs. (38)–(40), (47), and (45), one obtains the expression for the group velocity

$$\rho_X V_g^2=\frac{(1+q^2)[C_{11-}^2p^4-C_{33-}^2-D(D+2C_{33+}^{(3)})]^2}{8D^2(C_{11+}^{(1)}p^2+C_{33+}^{(3)}+D)}. \quad (52)$$

The above equation can be used in combination with either Eq. (49) or Eq. (51). The former case applies when the relevant C_{ij} and stresses acting in a medium are known. For a given group-velocity direction $q=\tan\zeta$, one uses Eq. (49) to find p; Eq. (32) to calculate D; and then Eq. (52) to obtain the group velocity. The latter case applies to an inverse approach by which C_{13+} can be found from the measured values of a group velocity V_g for the given direction q, $C_{11+}^{(1)}$, C_{11-}, $C_{33+}^{(3)}$, and C_{33-}. As indicated by Eq. (23), $C_{11+}^{(1)}$, $C_{11-}=C_{11-}^{(1)}$, $C_{33+}^{(3)}$, and $C_{33-}=C_{33-}^{(3)}$ can be obtained from measurements of the pure-mode wave speeds propagating in the symmetry directions. Equation (52), when D in it is substituted by Eq. (51) with the known values of q, V_g, $C_{11+}^{(1)}$, C_{11-}, $C_{33+}^{(3)}$, and C_{33-}, becomes a function of single variable p, which can be solved for to find the values of $D\geqslant 0$, B, and C_{13+} via Eqs. (51), (40), and (31), respectively. This in turn yields the value of C_{13} if the stresses σ_{11} and σ_{33} are known.

2. Quasitransverse mode

In the propagation of the QT mode, Eqs. (38) and (39) are both determined with the lower sign in front of D. The application of very similar procedures to those taken in the QL mode yields

$$q=\frac{p(B-C_{11-}^2p^2+C_{11+}^{(1)}D)}{Bp^2-C_{33-}^2+C_{33+}^{(3)}D}, \quad (53)$$

$$C_{11-}^2p^3+q(Bp^2-C_{33-}^2)-Bp-(C_{11+}^{(1)}p-C_{33+}^{(3)}q)D=0, \quad (54)$$

$$D=\frac{1}{1-pq}\{p(C_{11+}^{(1)}p-C_{33+}^{(3)}q)\pm[p^2(C_{11+}^{(1)}p-C_{33+}^{(3)}q)^2 -(1-p^2q^2)(C_{11-}^2p^4-C_{33-}^2)]^{1/2}\}, \quad (55)$$

$$\rho_x V_g^2=\frac{(1+q^2)[C_{11-}^2p^4-C_{33-}^2-D(D-2C_{33+}^{(3)})]^2}{8D^2(C_{11+}^{(1)}p^2+C_{33+}^{(3)}-D)}. \quad (56)$$

The calculation of the group velocity for a given direction ζ and the determination of C_{13+} from the relevant measurements can be achieved in a way similar to those achieved in the QL mode. However, the measurement of the group velocity of the QT mode is generally much more difficult than that of the QL mode in the signal generated by a small, pointlike source and detected by a small, pointlike piezoelectric detector, except in the case of the signal detected by the noncontact displacement transducer such as a capacitive transducer or a laser interferometer. Detailed discussion on the group velocity of the QL and QT modes and its application to determination of elastic constants in nonacoustoelastic case ($\sigma_{ij}=0$) is provided in Refs. 5–8.

C. Extension to higher-symmetry media

1. Stressed tetratropic medium

A material of cubic symmetry of 432, $\overline{43}m$, and $m3m$ classes behaves similar to but not exactly as one of tetragonal symmetry, when the material is stressed in three cubic-axes directions with two equal biaxial stresses or when it is uniaxially loaded along a cubic-axis direction. We term here such stressed media as having tetratropic symmetry. The Huang's condition Eq. (5) holds for tetratropic symmetry, while $\widetilde{C}_{ijkl}=\widetilde{C}_{klij}$ for tetragonal symmetry. This means that tetragonal symmetry is maintained only when hydrostatic pressures are applied to a material of tetragonal symmetry. A medium of tetragonal symmetry of 422, $4mm$, $\overline{4}2m$, and $4/mmm$ classes also behaves tetratropically when it is deformed with three principal stresses, $\sigma_{11}=\sigma_{22}$ and σ_{33}. It has a total of six thermodynamic elastic coefficients:

$$\begin{aligned}&C_{11}=C_{22},\quad C_{33},\quad C_{12},\\ &C_{13}=C_{23},\quad C_{44}=C_{55},\quad C_{66},\end{aligned} \quad (57)$$

where the direction of tetratropic symmetry is taken as the X_3 direction.

In a tetratropic material, the (010) plane is equivalent to the (100) plane and the group velocities of the PT, QL, and QT modes propagating in these planes and the (001) plane are governed by those equations previously derived for orthotropic material with the use of Eq. (57) and $\sigma_{11}=\sigma_{22}$. In the $\{n_1,\overline{n_1},0\}$-type diagonal symmetry plane, where the wave normal and group velocity propagate in the $\langle n_1,n_1,n_3\rangle$ direction at angles θ and ζ to the x_3 direction, respectively,

$$n_1^2=n_2^2=n_d^2/2=\sin^2\theta/2,\quad n_d^2+n_3^2=1. \quad (58)$$

Equation (9) is now factored as

$$(\Gamma_{11}-\Gamma_{12}-\rho V^2)[(\Gamma_{33}-\rho V^2)(\Gamma_{11}+\Gamma_{12}-\rho V^2)-2\Gamma_{13}^2] =0, \quad (59)$$

where

$$\begin{aligned}&\Gamma_{11}=(C_{11}^{(1)}+C_{66}^{(1)})n_d^2/2+C_{44}^{(3)}n_3^2,\\ &\Gamma_{33}=C_{44}^{(1)}n_d^2+C_{33}^{(3)}n_3^2,\\ &\Gamma_{12}=(C_{12}+C_{66})n_1n_2=(C_{12}^{(1)}+C_{66}^{(1)})n_d^2/2,\\ &\Gamma_{13}=(C_{13}+C_{44})n_1n_3=C_{13+}n_dn_3/\sqrt{2}.\end{aligned} \quad (60)$$

The first factor in parenthesis of Eq. (59) yields the relation for the PT mode propagating on the $\{n_1,\overline{n_1},0\}$-type plane and polarized in the $\langle n_1,\overline{n_1},0\rangle$ direction. Following the similar procedures as in Sec. III A, one finds exactly the same relations for the PT mode on the diagonal plane of a tetratropic medium by replacing $C_{66}^{(1)}$ by $(C_{11}^{(1)}-C_{12}^{(1)})/2$ in

Eqs. (20), (28a), (28b), and (29). The PT-mode group velocity is then given by

$$(\rho_x V_g^2)^{-1} = [(C_{11}^{(1)} - C_{12}^{(1)})/2]^{-1} \sin^2 \zeta + (C_{44}^{(3)})^{-1} \cos^2 \zeta \quad (PT \text{ mode}). \tag{61}$$

The terms in the square bracket of Eq. (59) yield relations for the QL and QT modes. Following the procedures similar to those in Sec. III B of this paper and Sec. II C of Ref. 5, one finds that exactly the same relations between V_g, $p = \tan\theta$, $q = \tan\zeta$, C_{ij}, and σ_{ij} can be obtained for the QL and QT modes in the corresponding Eqs. (46)–(56) and the defining Eqs. (31) and (32) by simply replacing: C_{55} by C_{44}; $C_{55}^{(1)}$ by $C_{44}^{(1)}$; $C_{55}^{(3)}$ by $C_{44}^{(3)}$; C_{11} by K; $C_{11\pm}$ by $K_\pm$; $C_{11}^{(1)}$ by $K^{(1)}$; and $C_{11\pm}^{(1)}$ by $K_\pm^{(1)}$, where K, $K_\pm$, $K^{(1)}$, and $K_\pm^{(1)}$ are now defined as

$$K \equiv (C_{11} + C_{12} + 2C_{66})/2, \tag{62a}$$

$$K_\pm \equiv K \pm C_{44}. \tag{62b}$$

$$K^{(1)} \equiv (C_{11}^{(1)} + C_{12}^{(1)} + 2C_{66}^{(1)})/2, \tag{63a}$$

$$K_\pm^{(1)} \equiv K^{(1)} \pm C_{44}^{(1)}, \tag{63b}$$

where $C_{11}^{(1)}$, $C_{66}^{(1)}$, $C_{12}^{(1)}$, and $C_{44}^{(1)}$ have previously been defined in Eqs. (17) and (18). The group-velocity and relevant relations for the QL and QT modes are

$$K_-^2 p^3 + q(Bp^2 - C_{33-}^2) - Bp \pm (K_+^{(1)} p - C_{33+}^{(3)} q) D = 0, \tag{64}$$

$$D = \frac{1}{1 - pq} \{ \pm p(C_{33+}^{(3)} q - K_+^{(1)} p) \pm [p^2 (C_{33+}^{(3)} q - K_+^{(1)} p)^2 - (1 - p^2 q^2)(K_-^2 p^4 - C_{33-}^2)]^{1/2} \}, \tag{65}$$

$$\rho_x V_g^2 = \frac{(1+q^2)[K_-^2 p^4 - C_{33-}^2 - D(D \pm 2C_{33+}^{(3)})]^2}{8D^2 (K_+^{(1)} p^2 + C_{33+}^{(3)} \pm D)}. \tag{66}$$

In Eqs. (64)–(66) above, the upper and lower signs in both $\pm$ and $\mp$ apply to the QL and QT modes, respectively, except for the $\pm$ sign in front of the square bracket in Eq. (65), which applies to both QL and QT modes. Note that B and D in Eqs. (64)–(66) are respectively defined by Eqs. (31) and (32), where $C_{11-} = C_{11-}^{(1)}$ is now replaced by $K_- = K_-^{(1)}$.

2. Stressed transversely isotropic medium

An isotropic medium at the natural state behaves as a transversely isotropic medium when it is loaded in three arbitrarily chosen, mutually perpendicular directions with two equal biaxial stresses, say $\sigma_{11} = \sigma_{22}$. The case of $\sigma_{11} = \sigma_{22} = 0$ is common in uniaxial tension or compression tests. In a transversely isotropic medium there are five independent thermodynamic elastic coefficients:

$$C_{11} = C_{22}, \quad C_{33}, \quad C_{12}, \quad C_{13} = C_{23}, \quad C_{44} = C_{55}, \quad C_{66} = (C_{11} - C_{12})/2. \tag{67}$$

Any plane parallel to the axis of transverse symmetry is called a zonal plane and all the zonal planes are identical. There is no distinction between the {010}- and {110}-type planes, which are all identical to the zonal plane. Any zonal plane is a principal plane and any direction normal to the zonal plane is also a principal-stress direction.

For the wave propagation in the zonal plane with wave normal **n** and group velocity $\mathbf{V}_g$ directed at angles θ and ζ to the X_3 symmetry axis, respectively, there is a one-to-one correspondence between the directions of the wave normal and the group velocity and all the points in the zonal slowness plane map themselves onto the same zonal group-velocity plane. What holds for the (010) plane of the orthotropic medium in Secs. III A and III B also holds for the zonal plane of a transversely isotropic medium with Eq. (67) and $\sigma_{11} = \sigma_{22}$ substituted in the appropriate relations.

For the wave propagation in the (001) basal plane normal to the axis of transverse symmetry, all the propagation directions are identical and principal symmetry directions. All the phase velocities are of the pure mode and coincide with group velocities of the pure mode. Their relations are indicated by the first-row elements of Eq. (23) with Eq. (67) and $\sigma_{11} = \sigma_{22}$ and satisfied.

IV. GROUP-VELOCITY SECTIONS ILLUSTRATED WITH A UNIAXIALLY LOADED SILICON CRYSTAL

In this section we illustrate the effect of stress on the group-velocity sections with a silicon crystal when it is uniaxially compressed in the direction that coincides with a cubic-axis direction of the crystal in the stress-free natural state. We will take, as an example, the (010) and (110) group-velocity sections when the silicon crystal is compressed normal to the (001) plane at $\sigma_{33} = -1$ GPa with all other stress components being zero. As mentioned in Sec. III C 1, the silicon crystal in this case behaves tetratropically and it has six thermodynamic elastic coefficients C_{ijkl} as indicated by Eq. (57). They are related to the second- and third-order elastic constants referred to the coordinates of the stress-free natural states by[12,22]

$$C_{ijkl} = \frac{\rho_X}{\rho_a} \frac{\partial X_i}{\partial a_p} \frac{\partial X_j}{\partial a_q} \frac{\partial X_k}{\partial a_r} \frac{\partial X_l}{\partial a_s} c_{pqrs}^N(\mathbf{X}) = \frac{\rho_X}{\rho_a} \frac{\partial X_i}{\partial a_p} \frac{\partial X_j}{\partial a_q} \frac{\partial X_k}{\partial a_r} \frac{\partial X_l}{\partial a_s} [c_{pqrs}(\mathbf{a}) + c_{pqrsmn}(\mathbf{a}) \eta_{mn} + \cdots], \tag{68}$$

where **a** denotes a coordinate of a particle in the stress-free natural state, $c_{pqrs}^N(\mathbf{X})$ is the thermodynamic elastic coefficient that is referred to the natural state and evaluated at the initial state **X**, $c_{pqrs}(\mathbf{a})$ and $c_{pqrsmn}(\mathbf{a})$ are the second-order and third-order elastic constants, which are both referred to and evaluated at the natural state **a**, and

$$\eta_{mn} = \left(\frac{1}{2}\right) \left[\frac{\partial u_m}{\partial a_n} + \frac{\partial u_n}{\partial a_m} + \frac{\partial u_s}{\partial a_m} \frac{\partial u_s}{\partial a_n} \right] \tag{69}$$

is a finite strain referred to the natural state.

$c_{pqrs}(\mathbf{a})$ for cubic silicon are:[2] $c_{11} = 165.7$ GPa, $c_{12} = 63.9$ GPa, and $c_{44} = 79.56$ GPa. Its density at the natural state is $\rho_a = 2332$ kg/m^3. Using these values of the second-order elastic constants and the density evaluated at the natural state, the (010) and (110) group-velocity sections of silicon are plotted with solid lines in Figs. 1 and 2, respectively.

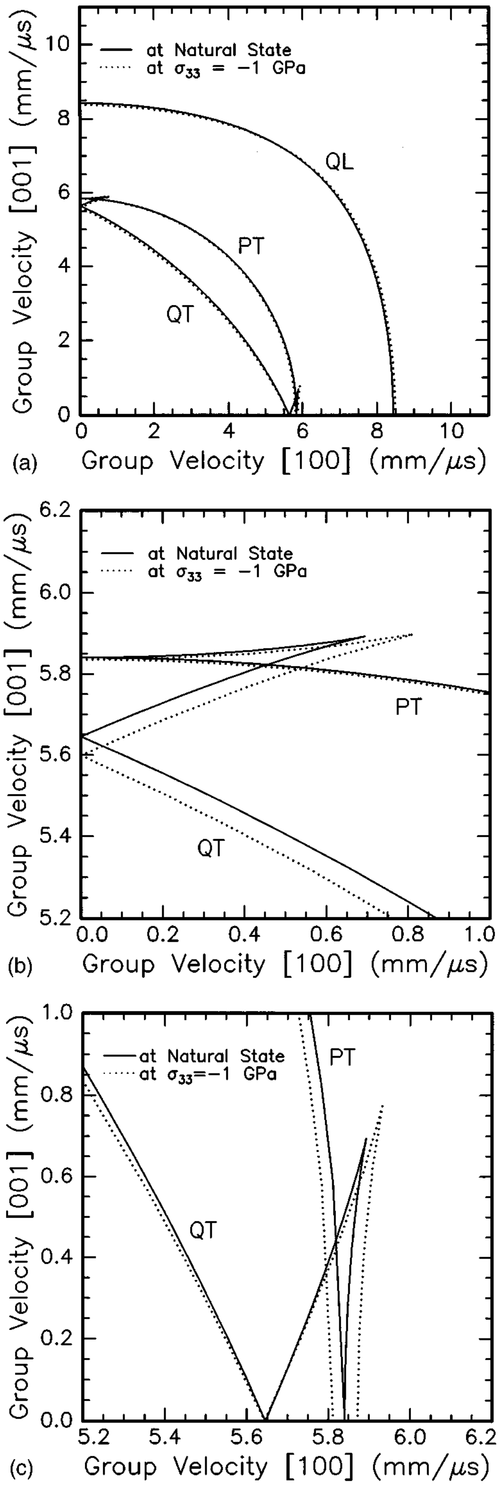

FIG. 1. The (010) group-velocity sections of silicon at natural state and at $\sigma_{33}=-1$ GPa: (a) global view; (b) expanded view of the transverse group-velocity sections near the [001] direction; and (c) expanded view of the transverse group-velocity sections near the [100] direction.

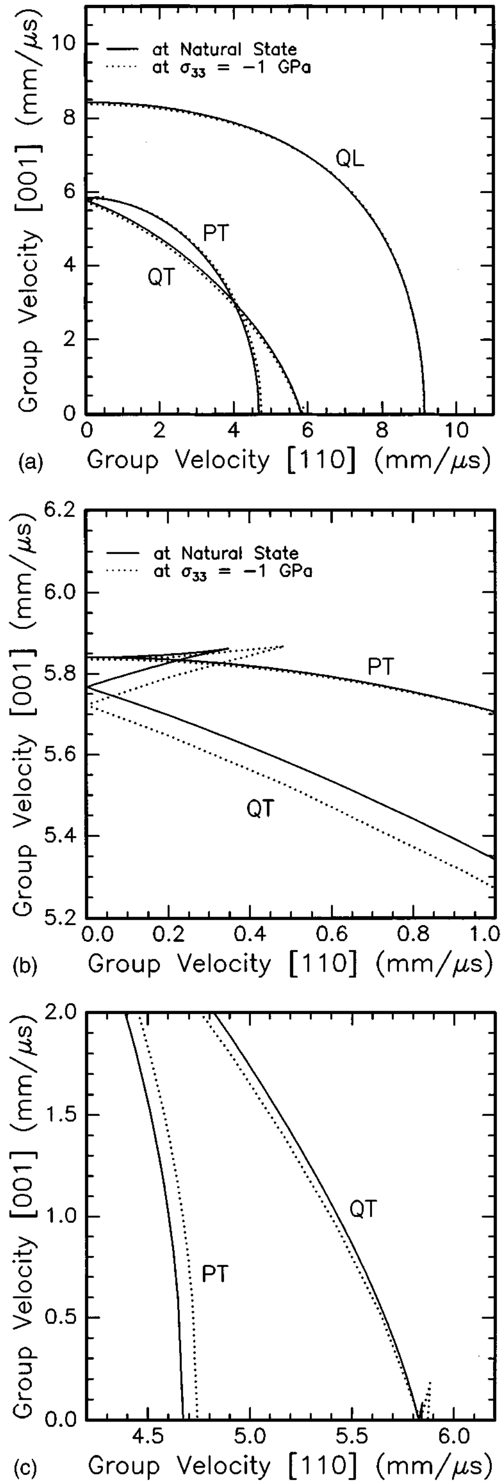

FIG. 2. The $(\bar{1}10)$ group-velocity sections of silicon at natural state and at $\sigma_{33}=-1$ GPa: (a) global view; (b) expanded view of the transverse group-velocity sections near the [001] direction; and (c) expanded view of the transverse group-velocity sections near the [110] direction.

The third-order elastic constants $c_{pqrsmn}(\mathbf{a})$ of silicon, evaluated in the natural state, are:[23,24] $c_{111}=-795$ GPa, $c_{112}=-445$ GPa, $c_{123}=-75$ GPa, $c_{144}=15$ GPa, $c_{155}=-310$ GPa, and $c_{456}=-86$ GPa. Using the values of the second-order and third-order elastic constants of silicon and the identity relations between the third-order elastic constants: $c_{111}=c_{222}=c_{333}$, $c_{144}=c_{255}=c_{366}$, $c_{112}=c_{223}=c_{133}=c_{113}=c_{122}=c_{233}$, $c_{155}=c_{244}=c_{344}=c_{166}=c_{266}=c_{355}$ for cubic silicon, C_{ijkl} at $\sigma_{33}=-1$ GPa are calculated according to Eq. (68) to yield: $C_{11}=C_{22}=168.51$ GPa, $C_{33}=165.31$ GPa, $C_{44}=C_{55}=80.70$ GPa, $C_{66}=79.06$ GPa, $C_{12}=63.32$ GPa, and $C_{13}=C_{23}=65.72$ GPa. The density $\rho_X=2340$ kg/m^3 is obtained at $\sigma_{33}=-1$ GPa. Using these thermodynamic elastic coefficients $C_{\mu\nu}$ obtained at $\sigma_{33}=-1$ GPa, the (010) and (110) group-velocity sections of silicon are also displayed in Figs. 1 and 2, respectively, with dotted lines. The group-velocity sections near the [001] and [100] axes of these symmetry planes, which correspond to the wave normals lying in nonsymmetry planes, are neither of interest here nor within the scope of this work and therefore not included in the figures. The group velocity of the longitudinal (L) mode in the [001] and [100] directions is 8.429 mm/μs in the natural state. Under $\sigma_{33}=-1$ GPa, the change in group velocity of this mode is about -0.59% in the [001] loading direction, while it is about 0.67% in the [100] transverse to the loading direction. The group velocity of the QL mode at $\sigma_{33}=-1$ GPa varies from that in the natural state by 0.37% in the [101] direction, 45° away from the loading direction, while it changes minimally by -0.016% in the [110] perpendicular to the loading direction. The group velocities along both the [101] and the [110] directions in the natural state are 9.129 mm/μs.

Figures 1(b), 1(c), 2(b), and 2(c) display detailed views of the group velocity sections of the transverse mode near the symmetry directions. Figures 1(b) and 2(b) indicate that along the [001] direction are four rays with distinct velocities: pure L ray shown, PT ray shown in both figures, intermediate-speed QT (IQT)ray shown in Fig. 2(b), and slow QT (SQT) ray shown in Fig. 1(b). These rays propagate at 8.429 mm/μs, 5.841 mm/μs, 5.767 mm/μs, and 5.646 mm/μs, respectively, in the natural state. Their speeds change respectively by -0.59% (already mentioned), -0.085%, -0.77%, and -0.82% at $\sigma_{33}=-1$ GPa. The group-velocity sections of the fast QT (FQT) and PT modes make a tangential contact with each other in the [001] direction, and their speeds are identical along this direction both in the natural state and at $\sigma_{33}=-1$ GPa. The FQT mode along the [001] direction becomes a pure mode which is polarized in the [100] direction, while the PT mode is polarized in the [010] direction as mentioned before. Note that the IQT and SQT rays are of the oblique mode, the wave normals of which lie in nonsymmetry directions on the (110) and (010) planes, respectively.

The IQT ray along the [100] direction in the natural state propagates at 5.767 mm/μs with its oblique wave normal lying on the {011}-type planes. However, this ray no longer travels along the [100] direction under the influence of stress σ_{33}, since the presence of the stress σ_{33} causes the {011}-type planes to be no longer symmetry planes. Referring to Fig. 1(c), the group velocity of the SQT ray with 5.646 mm/μs in the natural state varies very slowly with stress (-0.030% at $\sigma_{33}=-1$ GPa) along the [100] direction. The group velocities of the FQT and PT modes, which are degenerate with the speed 5.841 mm/μs along the [100] direction in natural state, split into opposite directions under the stress σ_{33} and the difference between them, 0.0602 mm/μs at $\sigma_{33}=-1$ GPa, is about 1.03% of their speed in the natural state. The former mode is polarized in the [001] direction, and the latter in the [010] as aforementioned. The group velocities of these pure modes along the [100] direction are equal to their phase velocities and they are respectively called the SV (shear vertical) and SH (shear-horizontal) types. Under a stress σ_{22} acting, these two modes interchange their role. It is well known in acoustoelasticity that the difference between the wave speeds of the SV and SH types that propagate in the [100] normal to the loading direction is proportional to the difference between the two principal stresses, $\sigma_{33}-\sigma_{22}$.[14]

In the face-diagonal direction [110] [see Fig. 2(c)], the group velocity of the FQT ray changes by 0.54% from 5.841 mm/μs in the natural state to 5.873 m when $\sigma_{33}=-1$ GPa, while that of the SQT ray hardly changes (0.15%) from 5.828 mm/μs in the natural state to 5.837 m at $\sigma_{33}=-1$ GPa. However, the change in the group velocity of the PT mode is quite significant. It varies by 1.5% from 4.672 mm/μs in the natural state to 4.741 mm/μs at $\sigma_{33}=-1$ GPa.

A phase velocity of the SH-polarized PT mode propagating in an oblique direction of the symmetry planes is difficult to measure, but its group velocity is easy to obtain by employing a point-source/point-detector technique.[7,8] This mode provides certain advantages for estimating residual stresses acting in an elastic body.[25,26] We take an example of the SH-mode propagating in the 45° direction to the loading direction in the (010) and (110) planes. It is easy to obtain from Eqs. (29) and (61) that these PT modes propagate in the natural state at the group velocities 5.841 mm/μs in the (010) plane and 5.160 mm/μs in the (110) plane, while at $\sigma_{33}=-1$ GPa they travel respectively at 5.824 mm/μs and 5.204 mm/μs. The effect of the uniaxial compressive stress σ_{33} on the PT mode in silicon is greater in the (110) plane than in the (010) plane.

It may be interesting to see the stress sensitivity of the magnitude and direction of group velocities at the cuspidal edges shown in Figs. 1(b), 1(c), 2(b), and 2(c). We follow the method adopted in Ref. 5 for calculation of the polar coordinates of these points, the magnitude of group velocity V_g and the angular direction ζ. In the natural state the coordinates of these cuspidal edges near the [001] direction shown in Figs. 1(b) and 2(b) are calculated to be (5.934 mm/μs, 6.72°) and (5.873 mm/μs, 3.40°). They respectively move to (5.953 mm/μs, 7.83°) and (5.887 mm/μs, 4.71°) at $\sigma_{33}=-1$ GPa. We notice that changes in the direction of the cuspidal edges are rather substantial, being more than 1°. The cuspidal edges near the [100] and [110] directions shown in Figs. 1(c) and 2(c) vary respectively from (5.934 mm/μs, 83.28°) and (5.846 mm/μs, 89.16°) to (5.983 mm/μs, 82.57°) and (5.886 mm/μs, 88.25°), resulting in the directional change of 0.71° and 0.91° and in the appreciable variation of the magnitude of their group velocities by 0.8% and 0.7%, respectively. However, because of the effect of *eidolon* associated with the diffraction of sound waves of finite wave length,[27] it is not easy to measure the directional change.

Overall, it is worth mentioning that the effect of stress on the group velocity of the PT and FQT modes is larger for waves propagating in the direction normal to loading and minimal along the loading direction, while for the oblique modes, such as IQT and SQT modes, the effect is the opposite; that is, much greater along the loading direction and minimal along the direction normal to loading. For the longitudinal mode propagating in the (010) plane, the effect is roughly equal in magnitude in both directions but opposite in sign. In the (110) plane the stress effect on the longitudinal mode is greater along the loading direction and minimal in the direction normal to the loading.

V. GROUP VELOCITY OF THE OBLIQUE MODE ALONG THE SYMMETRY DIRECTION

One distinct feature found in some anisotropic materials is the presence of an oblique-mode group velocities along the symmetry directions, such as the SQT and IQT rays shown in Figs. 1(b), 1(c), 2(b), and 2(c). These oblique-mode rays do not exist in an isotropic material. It may be worthwhile to derive the expressions governing the group velocity of the oblique modes along the symmetry directions. The oblique-mode rays are always associated with the QT slowness surface and the existence of these rays along the symmetry direction requires the QT slowness surface to be concave around that symmetry direction. The concavity condition in a stress-free natural state was discussed by Musgrave[1] and Wang[10] and we discuss the concavity condition of the QT slowness surface in a stressed medium for the case that only the normal stress components σ_{ii} (i not summed; $i=1,2,3$) are acting and all shear stress components in the medium are zero, i.e., $\sigma_{12}=\sigma_{13}=\sigma_{23}=0$. Following the similar procedures used in Refs. 1 and 10, it is easy to show the concavity conditions both of the (010) slowness section of the QT mode in an orthotropic material and of the $(1\bar{1}0)$ slowness section of the QT mode in a tetratropic material. For normal solids in which the conditions of $C_{11}>C_{55}$, $C_{33}>C_{55}$, and $K>C_{44}$ hold, the concavity conditions around the [001] direction are

$$C_{13+}^2 > C_{11}^{(1)} C_{33-} \tag{70a}$$

and

$$C_{13+}^2 > K^{(1)} C_{33-}, \tag{70b}$$

for the (010) and $(1\bar{1}0)$ QT slowness sections, respectively. Likewise, the concavity conditions around the [100] direction of the (010) QT slowness section and around the [110] direction of the (110) QT slowness section are respectively given by

$$C_{13+}^2 > C_{33}^{(3)} C_{11-} \tag{71a}$$

and

$$C_{13+}^2 > C_{33}^{(3)} K_{-}. \tag{71b}$$

When conditions (70a), (70b), (71a), and (71b) are satisfied, there exist the rays of the oblique mode propagating along the symmetry directions, and they are all satisfied in silicon. The expressions for the group velocity V_g of the oblique mode along the [001] direction can be derived by similar procedures found in Refs. 6 and 9 and written as

$$C_{11-}^2 (\rho_X V_g^2)^2 + 2(C_{11+}^{(1)} C_{13+}^2 - C_{11-} C_{pr-}) \rho_X V_g^2 + (C_{13+}^4 - 2C_{pr+} C_{13+}^2 + C_{pr-}^2) = 0 \tag{72}$$

for the ray whose oblique wave normal lies on the (010) plane of the orthotropic material, and

$$K_{-}^2 (\rho_X V_g^2)^2 + 2(K_{+}^{(1)} C_{13+}^2 - K_{-} K_{pr-}) \rho_X V_g^2 + (C_{13+}^4 - 2K_{pr+} C_{13+}^2 + K_{pr-}^2) = 0 \tag{73}$$

for the ray whose oblique wave normal lies on the $(1\bar{1}0)$ plane of the tetratropic material. In Eqs. (72) and (73) $C_{pr\pm}$ and $K_{pr\pm}$ are defined by

$$C_{pr\pm} \equiv C_{11}^{(1)} C_{33}^{(3)} \pm C_{55}^{(1)} C_{55}^{(3)}, \tag{74a}$$

$$K_{pr\pm} \equiv K^{(1)} C_{33}^{(3)} \pm C_{44}^{(1)} C_{44}^{(3)}. \tag{74b}$$

Similar expressions for the group velocity of the oblique mode propagating either in the [100] or in the [110] direction can be found by interchanging the indices 1 and 3 in Eq. (72) for the former direction and by interchanging K_{-} and C_{33-} and also interchanging $K_{+}^{(1)}$ and $C_{33+}^{(3)}$ in Eq. (73) for the latter direction. Using Eqs. (72) and (73) and equivalent equations, the group velocities of the IQT and SQT modes along the symmetry directions of silicon at $\sigma_{33}=-1$ GPa can be conveniently obtained and are identical with those in Figs. 1 and 2.

VI. DISCUSSION

We have derived various group-velocity formulas for the symmetry planes of a stressed elastic medium with orthotropic or higher symmetry and shown the effect of uniaxial compressive stress on the (010) and $(1\bar{1}0)$ group-velocity sections and the cuspidal features of a tetratropic silicon specimen as an example.

The derived formulas can also be used to determine the effective elastic coefficients and in principle the third-order elastic constants via Eq. (68). Since the determination of the third-order elastic constants are quite a complicated topic, interested readers are referred to Refs. 22 and 24 for detail. It was already mentioned that the pure-index effective elastic coefficients $C_{\mu\mu}^{(i)}$ (μ not summed; $\mu=1,2,...,6$; $i=1,2,3$) can be determined via Eq. (23) from measurements of the group or phase velocities of the pure-mode propagating in the symmetry direction. Determination of a mixed-index elastic coefficient, say C_{13+}, can be calculated from the QL or QT group-velocity data measured along an oblique direction in the symmetry plane and using the results developed in Sec. III. For detailed procedures on determination of the mixed-index elastic constants, readers are referred to Sec. III B of Ref. 5. If the QT slowness surface is concave around the symmetry direction, say the [001] axis, C_{13+} can be also determined via Eqs. (72) and (73) from measurements of the group velocities of the oblique mode along the [001] direction. Kim *et al.*[6,9] determined C_{13} of zinc and C_{12} of silicon in their stress-free, natural state from the measurements of the group velocities of the oblique mode propagating in the [001] direction. Since $C_{13+}=C_{13+}^{(1)}=C_{13+}^{(3)}$, one can also determine both $C_{13}^{(1)}=C_{13}-\sigma_{11}$ and $C_{13}^{(3)}=C_{13}-\sigma_{33}$, which appear in the expressions of the effective elastic coefficients $K_{ijkl}^{S} \equiv (\partial\sigma_{ij}/\partial\epsilon_{kl})_S$, the effective Young's modulus and Poisson's ratios.[18]

When isotropic materials, such as steel and aluminum alloy commonly used in engineering structures, are subjected to uniaxial tension or compression, they behave as transversely isotropic materials with the axis of transverse symmetry parallel to the loading direction. In this case Eq. (67) holds and one can calculate the group-velocity sections in a similar way as we did with the (001) silicon crystal. This situation often arises in estimation of residual stresses when it is necessary to measure the acoustoelastic birefringence constant. Since the third-order elastic constants of polycrystalline materials vary significantly from specimen to specimen and reliable, reported data of them are scarce in literature, we have not attempted here to predict the effects of stress on the group-velocity surfaces of structural materials.

When the applied stresses are hydrostatic pressures p such that $\sigma_{ij} = -p\,\delta_{ij}$, one can apply the superposition principle for the effect of three equal normal stresses acting on a medium to calculate the group velocities in the medium under hydrostatic pressures. The effect of hydrostatic pressures p on the elastic constants and the density of solids abounds in the literature.[24,28,29] Since the symmetry of a material remains unchanged under hydrostatic pressures, the effect of hydrostatic pressures on the group velocity in symmetry planes can be calculated following the same approaches taken in Ref. 5, whereby the elastic constants at the natural state are simply replaced by the effective elastic constants obtained under the hydrostatic pressures.

The effect of stress on the group velocities in nonsymmetry planes and even the group velocities on the symmetry planes, which correspond to the wave normals lying in a nonsymmetry plane, are very difficult to approach analytically. It may be calculated using the numerical method such as the Monte Carlo method. This is currently under investigation.

VII. CONCLUSIONS

We have derived closed-form analytic formulas that relate both thermodynamic elastic coefficients and stresses to the group velocities of PT, QL, and QT modes propagating in an arbitrary direction on the symmetry planes of a stressed medium with orthotropic or higher symmetry. Analytic formulas relating the directions of the group velocity and the corresponding wave normal on the symmetry plane are also presented for all three modes. These relations can be applied to determine the group-velocity sheets of the symmetry planes of the stressed medium and to obtain the mixed-index elastic coefficients. The group-velocity sections of silicon which is loaded in the [001] direction with a compressive stress of $\sigma_{33} = -1$ GPa are shown as a example.

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